Mathematics: analysis and approaches
Standard Level
Paper 2 (set C)

worked solutions

1. [Maximum mark: 6]

Let $f(x)=\dfrac{x+8}{x}$ and $g(x)=1-x^2$.

(a) Show that $f^{-1}(x)=\dfrac{8}{x-1}$. [3]

(b) (i) Write down $\left(f^{-1}\circ g\right)(x)$.

(ii) Solve the equation $\left(f^{-1}\circ g\right)(x)=x$. [3]

Solution:

(a) $y=\dfrac{x+8}{x} \Rightarrow x=\dfrac{y+8}{y} \Rightarrow xy=y+8 \Rightarrow xy-y=8 \Rightarrow y(x-1)=8$

Thus, $f^{-1}(x)=\dfrac{8}{x-1}$ ***QED***

(b) (i) $\left(f^{-1}\circ g\right)(x)=\dfrac{8}{\left(1-x^2\right)-1}=-\dfrac{8}{x^2}$

(ii) $-\dfrac{8}{x^2}=x \Rightarrow x^3=-8 \Rightarrow x=-2$

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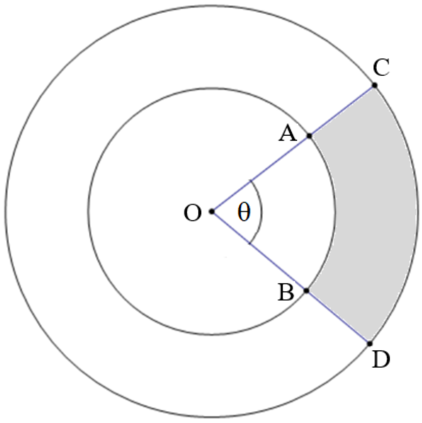
2. [Maximum mark: 6]

The diagram below shows two circles which have the same centre O. The smaller circle has a radius of 12 cm and the larger circle has a radius of 20 cm. The two arcs AB and CD have the same central angle θ, where $\theta=1.3$ radians. Find the area of the shaded region.



Solution:

area of shaded region = (area of sector COD) – (area of sector AOB)

area of sector COD $=\dfrac{1}{2}r^2\theta=\dfrac{1}{2}\cdot 20^2\cdot 1.3=260\text{ cm}^2$

area of sector AOB $=\dfrac{1}{2}r^2\theta=\dfrac{1}{2}\cdot 12^2\cdot 1.3=93.6\text{ cm}^2$

thus, area of shaded region $=260-93.6=166.4\approx 166\text{ cm}^2$

3. [Maximum mark: 5]

Find the coefficient of the x^3 term in the expansion of $\left(\frac{2}{3}x+3\right)^8$.

Solution:

general term of expansion: $\binom{8}{r}\left(\frac{2}{3}x\right)^{8-r}3^r$

considering exponent of x: $8-r=3 \Rightarrow r=5$

thus, coefficient of x^3 term $=\binom{8}{5}\left(\frac{2}{3}\right)^3 3^5 = 56\cdot\frac{8}{27}\cdot 243 = 4032$

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4. [Maximum mark: 6]

A car begins moving from a fixed point A. Its velocity, v ms^{-1}, after t seconds is given by $v=8-t^2-8e^{-t}$.

(a) Find the car's displacement from A when $t=4$. [3]

(b) Find the total distance that the car has travelled from A when $t=4$. [3]

Solution:

(a) displacement $=\int_0^4\left(8-t^2-8e^{-t}\right)dt \approx 2.81319...$

displacement from A when $t=4$ is approximately 2.81 meters

(b) total distance $=\int_0^4\left|8-t^2-8e^{-t}\right|dt \approx 12.34617...$

total distance from A when $t=4$ is approximately 12.3 meters

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5. [Maximum mark: 6]

The table below shows the marks earned on a quiz by a group of students.

Mark	1	2	3	4	5
Number of students	8	7	c	9	1

The median is 3 and the mode is 4 for the set of marks. Find the **three** possible values of c.

Solution:

Since 4 is the mode, then $c<9$

The total number of students is $8+7+c+9+1=c+25$

Since 3 is the median, then $\frac{c+25}{2}>8+7 \Rightarrow c+25>30 \Rightarrow c>5$

Thus, the three possible values of c are 6, 7, 8

6. [Maximum mark: 8]

Let $f(x) = x\ln\left(\frac{\mathrm{e}}{2x}\right)$. Point A is on the curve of f where $x = 1$. Point B is also on the curve of *f*. The tangent to the curve of f at A is perpendicular to the tangent at B. Find the coordinates of B.

Solution:

$$f'(x) = \ln\left(\frac{\mathrm{e}}{2x}\right) + x\left(\frac{2x}{\mathrm{e}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}}{2}x^{-1}\right)\right) = \ln\left(\frac{\mathrm{e}}{2x}\right) + x\left(\frac{2x}{\mathrm{e}}\left(-\frac{\mathrm{e}}{2x^2}\right)\right) = \ln\left(\frac{\mathrm{e}}{2x}\right) - 1 = \ln e - \ln(2x) - 1$$

Thus, $f'(x) = -\ln(2x)$

Gradient of tangent at A: $f'(1) = -\ln(2)$

Hence, gradient of tangent at B is $\frac{1}{\ln(2)}$

Find value of x such that the gradient of tangent to f is equal to $\frac{1}{\ln(2)}$.

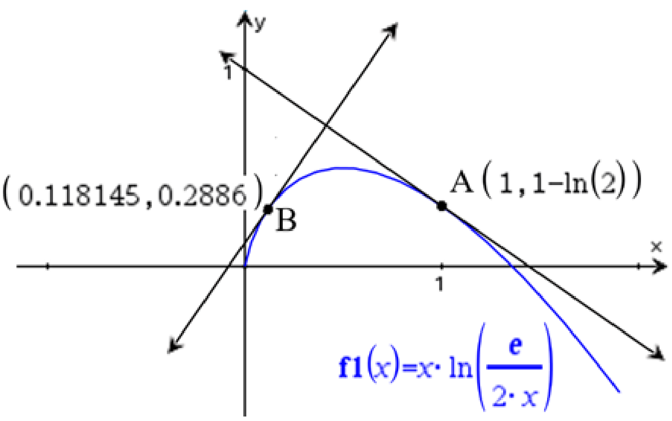
Solve $f'(x) = -\ln(2x) = \frac{1}{\ln(2)}$ $\quad\quad$ $\text{nSolve}\left(-\ln(2\cdot x) = \frac{1}{\ln(2)}, x\right)$ $\quad$ 0.118145044172

$f(0.118145044172...) \approx 0.288592313505...$

Thus, coordinates of B are $(0.118, 0.289)$

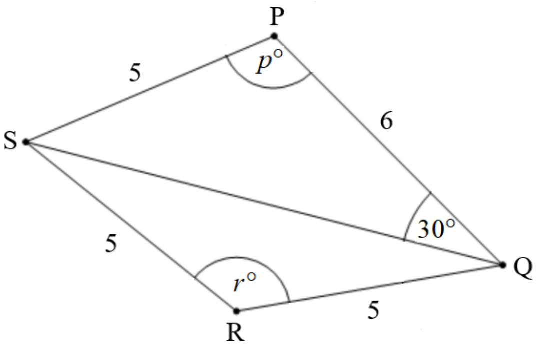
note: exact answer is $\mathrm{B}\left(\frac{1}{2e^{\frac{1}{\ln 2}}}, \frac{1+\ln 2}{2e^{\frac{1}{\ln 2}}\ln 2}\right)$, but this requires an unnecessary amount of work

Sketch illustrates that the tangents at A and B appear to be perpendicular



7. [Maximum mark: 15]

The diagram shows the quadrilateral PQRS.
Angle QPS and angle QRS are obtuse.



$\text{PQ} = 6\,\text{cm},\ \text{QR} = 5\,\text{cm},\ \text{RS} = 5\,\text{cm},\ \text{PS} = 5\,\text{cm},\ \text{P}\hat{\text{Q}}\text{S} = 30°$,
$\text{Q}\hat{\text{P}}\text{S} = p°,\ \text{Q}\hat{\text{R}}\text{S} = r°$

(a) Use the sine rule to show that $\text{QS} = 10\sin p$. [1]

(b) Use the cosine rule in triangle PQS to find another expression for QS. [3]

(c) (i) Hence, find p, giving your answer to two decimal places.

(ii) Find QS. [6]

(d) (i) Find r.

(ii) Hence, or otherwise, find the area of triangle QRS. [5]

Solution:

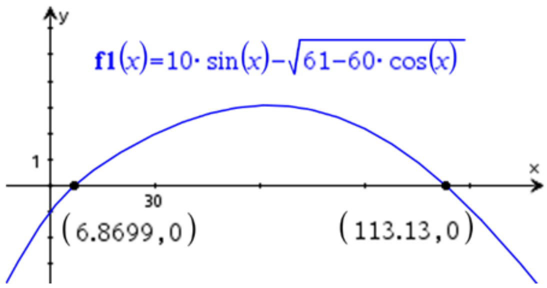
(a) $\dfrac{\sin 30°}{5} = \dfrac{\sin p}{\text{QS}} \Rightarrow \text{QS} = \dfrac{5}{1/2}\sin p \Rightarrow \text{QS} = 10\sin p$ ***QED***

(b) $\text{QS}^2 = 5^2 + 6^2 - 2(5)(6)\cos p \Rightarrow \text{QS} = \sqrt{61 - 60\cos p}$

(c) (i) solve the equation: $10\sin p = \sqrt{61 - 60\cos p}$

clearly, $90 < p < 180$; solve with graph or GDC solver

$\text{nSolve}\left(10\cdot\sin(p) = \sqrt{61-60\cdot\cos(p)}, p\right)|p>6.87$
113.130102354



thus, $p \approx 113.13$

(c) (ii) $\text{QS} = \sqrt{61 - 60\cos(113.130102354°)} \approx 9.19615\ldots \Rightarrow \text{QS} \approx 9.20\ \text{cm}$

(d) (i) using cosine rule in triangle QRS

$$(9.19615\ldots)^2 = 5^2 + 5^2 - 2(5)(5)\cos r \Rightarrow r = \cos^{-1}\left(\frac{50 - (9.19615\ldots)^2}{50}\right) \approx 133.7398\ldots \Rightarrow r \approx 134$$

(ii) area of triangle $\text{QRS} = \dfrac{1}{2}(5)(5)\sin(133.7398\ldots°) \approx 9.0310889\ldots \approx 9.03\ \text{cm}^2$

8. [Maximum mark: 14]

A commercial plantation grows pineapples that are classified as small, medium or large. The masses of the pineapples harvested in the year 2021 were normally distributed with a mean of 900 grams.

A pineapple is small if its mass is less than 750 grams. For the year 2021, sixteen percent of the pineapples are classified as small.

(a) Find the standard deviation of the masses of the pineapples. [4]

The following table shows the percentages of small, medium and large pineapples grown on the plantation in 2021.

small	medium	large
16%	63%	21%

(b) Given that a pineapple is classified as medium if its mass is greater than 750 grams and less than *m* grams, find the value of *m*. [2]

The plantation sends a shipment containing all of its medium and large pineapples harvested in 2021 to a food distribution company.

(c) A pineapple is randomly selected from this shipment. Find the probability that the pineapple is classified as medium. [3]

(d) The food distribution company sells all the pineapples in the shipment. It sells each of the medium pineapples for \$3.30 and each of the large pineapples for \$4.10. The food distribution company paid \$900 for the shipment and makes a profit of \$500 after selling all the pineapples in the shipment. Find the total number of pineapples in the shipment. [5]

<u>Solution</u>:

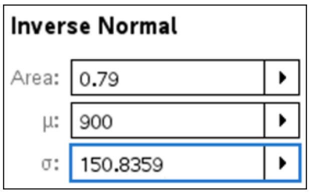
(a) $X \sim \mathrm{N}(900, \sigma)$ and $\mathrm{P}(X < 750) = 0.16 \Rightarrow Z \approx -0.994458...$

$$Z = \frac{x-\mu}{\sigma} \Rightarrow -0.994458... = \frac{750-900}{\sigma} \Rightarrow \sigma = \frac{-150}{-0.99446...} \approx 150.836... \Rightarrow \sigma \approx 151 \text{ grams}$$

(b) $\mathrm{P}(\text{small} + \text{medium} < m) = 0.16 + 0.63 = 0.79 \Rightarrow Z \approx 0.806421...$

$$0.80642... = \frac{x-900}{150.836...} \Rightarrow x = (0.80642...)(150.836...) + 900 \approx 1021.64... \Rightarrow m \approx 1020 \text{ grams}$$

OR, in one step on GDC:



$\mathrm{invNorm}(0.79, 900, 150.8359)$

1021.63727513

(c) Find probability of selecting a medium pineapple given it's in the shipment of medium & large

$$\mathrm{P}(\text{medium} \mid \text{in shipment}) = \frac{\mathrm{P}(\text{medium} \cap \text{in shipment})}{\mathrm{P}(\text{in shipment})} = \frac{0.63}{0.63+0.21} = \frac{0.63}{0.84} = 0.75 \text{ or } \frac{3}{4}$$

(d) $\mathrm{P}(\text{large}) = 1 - 0.75 = 0.25$ total sales – 900 = 500 → total sales = 1400

Let *x* be the number of pineapples in the shipment:

$$3.30(0.75x) + 4.10(0.25x) = 1400 \Rightarrow 3.5x = 1400 \Rightarrow x = \frac{1400}{3.5} = 400 \text{ pineapples in shipment}$$

9. [Maximum mark: 14]

(a) Given that $h(x)=\dfrac{ax-1}{bx-b}$, find the equation of the vertical asymptote and the equation for the horizontal asymptote for the graph of h. [2]

The vertical and horizontal asymptotes for the graph of h intersect at the point F.

(b) Write down the coordinates of F. [2]

(c) The point $\mathrm{G}(x,y)$ lies on the graph of h. Show that $\mathrm{FG}=\sqrt{(x-1)^2+\left(\dfrac{a-1}{bx-b}\right)^2}$. [4]

(d) Hence, find the coordinates of the points on the graph of $y=\dfrac{4x-1}{2x-2}$ that are closest to the point $(1,2)$. [6]

Solution:

(a) vertical asymptote: $x=1$, horizontal asymptote: $y=\dfrac{a}{b}$

(b) $\mathrm{F}\left(1,\dfrac{a}{b}\right)$

(c) applying distance formula for distance between $\mathrm{F}\left(1,\dfrac{a}{b}\right)$ and $\mathrm{G}(x,y)$

$$\mathrm{FG}=\sqrt{(x-1)^2+\left(y-\frac{a}{b}\right)^2} \quad \text{substitute } \frac{ax-1}{bx-b} \text{ for } y$$

$$\mathrm{FG}=\sqrt{(x-1)^2+\left(\frac{ax-1}{bx-b}-\frac{a}{b}\right)^2}=\sqrt{(x-1)^2+\left(\frac{ax-1}{b(x-1)}-\frac{a(x-1)}{b(x-1)}\right)^2}$$

$$=\sqrt{(x-1)^2+\left(\frac{ax-1-ax+a}{bx-b}\right)^2}=\sqrt{(x-1)^2+\left(\frac{a-1}{bx-b}\right)^2} \quad \textbf{QED}$$

(d) $y=\dfrac{4x-1}{2x-2} \Rightarrow a=4, b=2 \Rightarrow$ F is at $(1,2)$; thus, the distance from $(1,2)$ to any point $\mathrm{G}(x,y)$ on the graph of h is given by $\mathrm{FG}=\sqrt{(x-1)^2+\left(\dfrac{3}{2x-2}\right)^2}$

graph of expression for FG shows minimum occurs at $x\approx-0.22474...$ and $x\approx 2.22474...$

$y(-0.22474..)\approx 0.7753..,\quad y(2.22474..)\approx 3.2247..$

Thus, coordinates of the 2 points on $y=\dfrac{4x-1}{2x-2}$ that are closest to $(1,2)$ are

$(-0.225, 0.775)$ and $(2.22, 3.22)$